ROTATION OF A CYLINDER WITH A VARIABLE ANGULAR VELOCITY IN A VISCO-PLASTIC MEDIUM

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The paper contains the formulation and the solution of the problem on non-steady flow of a visco-plastic medium, surrounding a cylinder which rotates with a variable angular velocity. A method is developed to solve plane, axially symmetric "problems with a sought boundary" for parabolic equations. An equation is found for the determination of the radius of propagation of visco-plastic flow.

1. Formulation of the problem. Let a rigid cylinder of radius R be placed into a visco-plastic medium, occupying all space, and let it rotate in this medium with an angular velocity $\omega = \omega(t)$. We shall study the motion of the medium adjoining the cylinder. By contrast to a viscous fluid, the flow will extend only a finite distance from the rotating cylinder while the remaining part of the medium will be at rest. The radius of the zone of the visco-plastic flow depends on the properties of the medium and the angular velocity of rotation of the cylinder, and is a function of time, *a priori* unknown.

Due to symmetry the flow is described by a single equation

$$\rho \frac{\partial v}{\partial t} = \mu \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2}\right) - \frac{2\tau_0}{r} \qquad (1.1)$$

for $t > 0$, $R < r < r_0(t)$, where v is the tangential component of the velocity, ρ is the density, μ is the coefficient of viscosity, τ_0 is the limiting shear stress. The boundary and initial conditions are

$$v(r, 0) = F(r) \qquad \text{for } R < r < r_0(0) \qquad (1.2)$$

$$v(R, t) = R\omega(t) \qquad (1.3)$$

$$v(r_0(t), t) = 0, \qquad v_r(r_0(t), t) = 0 \qquad (1.4)$$

for $t > 0$. The conditions (1.4) express the vanishing of the velocities of rotation and slip on the boundary of the visco-plastic flow, whose radius is designated by $r_0(t)$ and is to be determined. Such general formulation of the problem may be found in [1].

Let ω_0 be a characteristic angular velocity and ν the coefficient of kinematic viscosity.

Let us introduce a dimensionless radius, a time and an angular velocity by the formulas

$$x = \frac{r}{R}, \qquad y = \frac{\nu}{R^2} t, \qquad W(y) = \frac{\omega(t)}{\omega_0}$$

and let us reduce the equation of motion and the boundary conditions to the dimensionless form

$$\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial x^2} + \frac{1}{x}\frac{\partial u}{\partial x} - \frac{u}{x^2} - \frac{2S}{x} \qquad \left(\begin{matrix} y > 0 \\ 1 < x < \delta(y) \end{matrix}\right) \tag{1.5}$$

Here $u(x, y)$ is the dimensionless velocity, $\delta(y)$ is the dimensionless radius of the zone of visco-plastic flow, S is the Saint-Venant parameter:

$$u(x, y) = \frac{v(r, t)}{R\omega_0}, \qquad \delta(y) = \frac{r_0(t)}{R}, \qquad S = \frac{\tau_0}{\mu\omega_0} \tag{1.6}$$

The boundary conditions (in the following they will be understood as the limiting ones) take on the form:

$$\lim u(x, y) = \frac{F(Rx)}{R\omega_0} = \Phi(x) \quad \text{for } y \to +0, \ (1 < x < \delta_0), \ \left(\delta_0 = \frac{r_0(0)}{R}\right) \tag{1.7}$$

$$\lim u(x, y) = w(y) \qquad \text{for } x \to 1 + 0, \ y > 0 \tag{1.8}$$

$$\lim u(x, y) = 0 \qquad \text{for } x \to \delta(y) - 0, \ y > 0 \tag{1.9}$$

$$\lim \frac{\partial u(x, y)}{\partial x} = 0 \qquad \text{for } x \to \delta(y) - 0, \ y > 0 \tag{1.10}$$

Here δ_0 is the initial radius of the zone of visco-plastic flow. The formulated boundary value problem is a typical "problem with sought boundary" for an equation of the parabolic type. One of the most effective methods of solution of such problems is the method of Kolodner [2], suggested by him for linear problems on phase changes. This method permits to find the unknown boundary of the region without constructing a solution within the region itself. An analogous method for plane axially-symmetric problems is developed in the present paper.

2. Construction of solution. We shall seek the solution of equation (1.5) in the form

$$u(x, y) = -\frac{2S}{x} y + \frac{1}{2y} \int_1^{\delta_0} \xi \Phi(\xi) I_1 \left(\frac{x\xi}{2y}\right) \exp\left(-\frac{x^2 + \xi^2}{4y}\right) d\xi + \lambda(x, y) \qquad (2.1)$$

As is easily verified, the second term satisfies for all x and $y > 0$ the equation

$$\frac{\partial}{\partial y} - \frac{\partial^2}{\partial x^2} - \frac{1}{x} \frac{\partial}{\partial x} + \frac{1}{x^2} = 0$$

Let us show that

$$\lim_{y \to +0} \frac{1}{2y} \int_1^{\delta_0} \xi \Phi(\xi) I_1 \left(\frac{x\xi}{2y}\right) \exp\left(-\frac{x^2 + \xi^2}{4y}\right) d\xi = \Phi(x) \qquad (1 < x < \delta_0)$$

Noting that for large arguments

$$I_1(z) \sim \frac{1}{\sqrt{2\pi z}} e^z \qquad (2.2)$$

and introducing a new variable of integration $\alpha = (\xi - x)/2\sqrt{y}$, we obtain

$$\lim_{y \to +0} \frac{1}{2y} \int_1^{\delta_0} \xi \Phi(\xi) I_1 \left(\frac{x\xi}{2y}\right) \exp\left(-\frac{x^2 + \xi^2}{4y}\right) d\xi =$$

$$= \frac{1}{\sqrt{\pi x}} \lim_{y \to +0} \int_1^{\delta_0} \sqrt{\xi} \Phi(\xi) \exp\left(-\frac{(x - \xi)^2}{4y}\right) \frac{d\xi}{2\sqrt{y}} =$$

$$= \frac{1}{\sqrt{\pi x}} \lim_{y \to +0} \int_{\chi_2}^{\chi_1} \sqrt{x + 2\alpha\sqrt{y}}\, \Phi(x + 2\alpha\sqrt{y}) e^{-\alpha^2} d\alpha = \Phi(x)$$

$$\chi_1 = (\delta_0 - x)/2\sqrt{y}, \quad \chi_2 = (1 - x)/2\sqrt{y} \qquad (1 < x < \delta_0)$$

The passage to the limit under the integral sign is permitted, since the function $\Phi(x)$ is assumed to be continuous. At the end of the interval the limit depends on the path of approach to the points $M_1(1, 0)$ and $M_2(\delta_0, 0)$. If the approach is along the straight lines $x = 1$ and $x = \delta_0$, this limit equals $1/2\, \Phi(1) = 1/2\, \Phi(\delta_0) = 0$.

For the function $\lambda(x, y)$ we will have the following boundary value problem

$$\frac{\partial \lambda}{\partial y} = \frac{\partial^2 \lambda}{\partial x^2} + \frac{1}{x} \frac{\partial \lambda}{\partial x} - \frac{\lambda}{x^2} \qquad (y > 0,\ 1 < x < \delta(y)) \qquad (2.3)$$

$$\lim \lambda(x, y) = 0 \qquad \text{for} \quad y \to +0,\ 1 < x < \delta_0 \qquad (2.4)$$

$$\lim \lambda(x, y) = W(y) + 2Sy - \frac{1}{2y}\int_1^{\delta_0} \xi\Phi(\xi) I_1\left(\frac{\xi}{2y}\right)\exp\left(-\frac{1+\xi^2}{4y}\right) d\xi = f(y)$$

$$\text{for } x \to 1+0 \tag{2.5}$$

$$\lim \lambda(x, y) = \frac{2Sy}{\delta(y)} - \frac{1}{2y}\int^{\delta_0} \xi\Phi(\xi) I_1\left(\frac{\xi\delta(y)}{2y}\right) \exp\left(-\frac{\xi^2+\delta^2(y)}{4y}\right) d\xi = \varphi(y)$$

$$(\text{for } x \to \delta(y) - 0 \text{ and } y > 0) \tag{2.6}$$

$$\lim \frac{\partial\lambda(x, y)}{\partial x} = -\frac{2Sy}{\delta^2(y)} - \frac{1}{4y^2}\int_1^{\delta_0} \xi\Phi(\xi)\left\{\xi I_1'\left(\frac{\xi\delta(y)}{2y}\right) - \right.$$

$$\left. - \delta(y) I_1\left(\frac{\xi\delta(y)}{2y}\right)\right\}\exp\left(-\frac{\xi^2+\delta^2(y)}{4y}\right) d\xi = \Psi(y)$$

$$(\text{for } x \to \delta(y) - 0 \text{ and } y > 0) \tag{2.7}$$

Let us delimit the region of variation of y; then the region in which the solution will be sought will be the region $D_-\{0 < y \leqslant y_0,\ 1 < x < \delta(y)\}$. Let $D_+\{0 < y \leqslant y_0,\ \delta(y) < x < \infty\}$ be the supplementary region of D_-. Let us extend the determination of the solution into the region D_+, assuming that $\lambda \equiv 0$ for $Q \in D_+$, then the boundary conditions for λ remain the same. Let us denote by $\bar{D}$ the closure of D_- and D_+ in the set $E\{0 < y < y_0,\ 1 < x < \infty,\ |x - \delta_0| + |y| > 0\}$, and by D the interior of D. Obviously, neither D, nor D depend on $\delta(y)$. The quantities $\lambda_-(x, y)$ and $\lambda_+(x, y)$ will be determined as $\lambda_\pm(M) = \lim \lambda(Q)$ for $Q \to M(x, y)$, where $Q \in D_\pm$.

Let us seek the solution of the equation (2.3) in the region D, satisfying the conditions (2.4) and (2.5) and two jump conditions on an arbitrary curve $x = \delta(y)$:

$$\lim_{x\to\delta(y)+0} \lambda(x, y) - \lim_{x\to\delta(y)-0} \lambda(x, y) = -\varphi(y) \tag{2.8}$$

$$\lim_{x\to\delta(y)+0} \frac{\partial\lambda(x, y)}{\partial x} - \lim_{x\to\delta(y)-0} \frac{\partial\lambda(x, y)}{\partial x} = -\Psi(y) \tag{2.9}$$

Furthermore, we require that the conditions would satisfy

$$\lambda(\infty, y) = 0 \tag{2.10}$$

and that the constructed solution be bounded together with its derivative in the region $\bar{D}$. It will be shown below that this problem has a solution and that it is unique.

The solution will be obtained in the form of a sum of a regular solution $\Theta(x, y)$ and a non-regular one $N(x, y)$. We shall require that the function $\Theta(x, y)$ satisfy the zero initial condition and the conditions

$$\lim \Theta(x, y) = f(y) - N(1, y) \quad \text{for } x \to 1+0, \Theta(\infty, \quad y) = 0 \tag{2.11}$$

and the function $N(x, y)$ in additon to conditions (2.4), (2.8), (2.9) and (2.10), also satisfy the condition $N(0, y) = 0$; then the sum of the solutions will satisfy all the conditions of the formulated problem.

The regular solution $\Theta(x, y)$ may be understood as the distribution of velocities in the viscous fluid for a rotating cylinder with an angular velocity $\alpha(y) = f(y) - N(1, y)$. This solution may be represented in the form [3]

$$\Theta(x, y) = \frac{\alpha(y)}{x} + \frac{2}{\pi}\alpha(0)\int_0^\infty \frac{J_1(\rho x) N_1(\rho) - J_1(\rho) N_1(\rho x)}{J_1^2(\rho) + N_1^2(\rho)} \frac{e^{-\rho^2 y}}{\rho} d\rho +$$
$$+ \frac{2}{\pi}\int_0^\infty \frac{J_1(\rho x) N_1(\rho) - J_1(\rho) N_1(\rho x)}{J_1^2(\rho) + N_1^2(\rho)} \frac{e^{-\rho^2 y}}{\rho} \int_0^y \alpha'(\eta) e^{+\rho^2\eta} d\eta d\rho \tag{2.12}$$

where $J_1(\rho)$ and $N_1(\rho)$ are Bessel and Neumann functions of the first order. The non-regular solution will be sought in the form of a sum of integrals of the type of potentials of simple and double layers of heat sources, uniformly distributed on the circle of radius $x = \delta(y)$. The strengths of the sources are selected such that conditions (2.8) and (2.9) be satisfied. Let us show that the unique solution will be

$$N(x, y) = -\frac{1}{4}\int_0^y G(\eta) d\eta + \frac{1}{2}\int_0^y H(\eta) d\eta + \tag{2.13}$$
$$+ \frac{1}{4}\int_0^y \frac{2\psi(\eta)\delta(\eta) + \varphi(\eta)}{y-\eta} I_1\left(\frac{x\delta(\eta)}{2(y-\eta)}\right) \exp\left(-\frac{x^2+\delta^2(\eta)}{4(y-\eta)}\right) d\eta$$

Here

$$G(\eta) = \frac{\varphi(\eta)\delta(\eta)}{(y-\eta)^2}\left\{xI_0\left(\frac{x\delta(\eta)}{2(y-\eta)}\right) - \delta(\eta) I_1\left(\frac{x\delta(\eta)}{2(y-\eta)}\right)\right\} \exp\left(-\frac{x^2+\delta^2(\eta)}{4(y-\eta)}\right) \tag{2.14}$$

$$H(\eta) = \frac{\varphi(\eta)\delta(\eta)\delta'(\eta)}{y-\eta} I_1\left(\frac{x\delta(\eta)}{2(y-\eta)}\right) \exp\left(-\frac{x^2+\delta^2(\eta)}{4(y-\eta)}\right)$$

It is easily verified that (2.13) for $x \neq \delta(y)$ satisfies equation (2.3), the initial condition (2.4) and the conditions for $x = 0$ and $x = \infty$. Let us prove that the conditions (2.8) and (2.9) are also satisfied. Let us designate the first two terms by K_1 and the third by K_2, and let us investigate their properties.

(a) *Properties of $K_1(x, y)$.* We split the interval of integration into two: one from 0 to $y - \epsilon$ and the other from $y - \epsilon$ to y. Then K_1 can be represented as a sum of integrals, and having selected $\epsilon > 0$ sufficiently small and employed asymptotic expansion or a Bessel function of a large argument, we obtain

$$-K_1(x, y) = \frac{1}{4}\int_0^{y-\varepsilon} G(\eta)\,d\eta - \frac{1}{2}\int_0^{y-\varepsilon} H(\eta)\,d\eta +$$

$$+\frac{1}{\sqrt{\pi x}}\int_{y-\varepsilon}^{y} \varphi(\eta)\sqrt{\delta(\eta)}\,\frac{\partial}{\partial\eta}\left\{\frac{x-\delta(\eta)}{2(y-\eta)^{1/2}}\right\}\exp\left(-\frac{[x-\delta(\eta)]^2}{4(y-\eta)}\right)d\eta \qquad (2.15)$$

The first two terms have no singularities and are continuous differentiable functions. The third term represents a combination of linear thermal potentials of simple and double layers. In paper [4] it was shown that such a combination has a discontinuity on the curve $x = \delta(y)$, and the jump equals $\phi(y)$. It is assumed thereby that the function $\delta(y)$ is continuously differentiable, is never-vanishing, and that $\delta'(y) \leqslant C/\sqrt{y}$. The function $\Phi(x)$ is assumed to be continuously differentiable and its derivative satisfying the Lipshitz condition in the interval $(1 < x < \delta_0)$. Calculating the derivative $\partial K_1/\partial x$, we find that it has a jump on the curve $x = \delta(y)$, which equals $\phi(y)/2\delta(y)$.

(b) *Properties of $K_2(x, y)$.* The function K is represented as an integral of the type of a potential of a simple heat layer and is therefore continuous. Its derivative, within a factor, is expressed by a linear thermal potential of a double layer, which has a discontinuity on the curve $x = \delta(y)$. Carrying out calculations we find that the jump is

$$\left[\frac{\partial K_2}{\partial x}\right] = -\left(\psi(y) + \frac{\varphi(y)}{2\delta(y)}\right)$$

From the properties of K_1 and K_2 it follows that their sum satisfies the conditions (2.8) and (2.9). The proof of uniqueness of the obtained equation is the same as in paper [4].

If we require that

$$\lim \lambda(x, y) = 0 \quad \text{for } x \to \delta(y) + 0, \qquad \lim \frac{\partial\lambda(x, y)}{\partial x} = 0 \quad \text{for } x \to \delta(y) + 0 \qquad (2.16)$$

then conditions (2.6) and (2.7) will be satisfied. This concludes the proof that $\lambda(x, y) = \theta(x, y) + N(x, y)$ is the unique solution of the boundary value problem (2.3) to (2.7). But the constructed solution contains the so far arbitrary function $\delta(y)$. For its determination we have two equations (2.16). It may be shown that an arbitrary solution of the first equation (2.16) also satisfies the second equation of (2.16) and vice versa (for proofs the reader is referred to the papers [2] or [4]).

Thus, if at least one of the equations (2.16) has the unique solution for $\delta(0) = \delta_0$, then it is this solution which yields the sought radius of the extent of visco-plastic flow, while (2.1) gives the distribution of velocities of this flow. We note that the question of uniqueness of the solutions of equations (2.16) has not as yet been resolved in a general form. For a series of concrete problems such proof may be found in the literature. An estimate of the constructed solution "in the large", that is for $0 < y < \infty$, may be given, but to this end supplementary investigations are required, which we have not presented here.

(c) *The case of a bounded medium.* In the preceding section we have constructed the solution for the case of an unbounded medium. If the visco-plastic medium has finite dimensions, the problem is somewhat more complicated. Let us consider the most simple case which will serve to clarify the mechanism of the flow propagation. Let the cylinder of the radius R be placed in the visco-plastic medium, whose boundary is of radius R_1, and let the cylinder start to rotate from rest with constant angular velocity $\omega =$ const. Obviously, a non-steady flow of the material is induced, which asymptotically approaches to a steady state as $y \to \infty$. The distribution of velocities for steady state flow is given by

$$v = \frac{\tau_0}{2\mu} r \left(\frac{\rho^2}{r^2} - 1 + 2 \ln \frac{r}{\rho}\right) \tag{3.1}$$

and the radius of the zone of the extent of flow ρ is given by the solution of the transcendental equation

$$\left(\frac{\rho}{R}\right)^2 - \ln\left(\frac{\rho}{R}\right)^2 = 1 + \frac{2\mu\omega}{\tau_0} \tag{3.2}$$

If $\rho \leqslant R$, then the boundary of the medium cannot influence the evolution of flow. Its influence becomes noticeable only for $\rho > R_1$. Up to the instant for which the flow does not reach the boundary of the medium the distribution of velocities obeys the same law as in the case of the unbounded medium, and commencing with this instant the law of flow will be changed. To find this law a new boundary value problem must be solved, omitting the condition of vanishing slip velocities at the boundary of the medium. Such a solution is given in [1]. However, the author does not take into account either the influence of the dimensions of the medium or the regions of validity of the solution obtained by him. In constructing the solution the author assumes that the whole medium is brought into rotation instantaneously; in such an approach the fundamental problem regarding the evolution of the zone of flow and of the determination of its radius remains unclarified.

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BIBLIOGRAPHY

1. Bakhshiian, F.A., Vrashchenie zhestkogo tsilindra v viazko-plastichnoi srede (Rotation of a rigid cylinder in a visco-plastic medium). *PMM* Vol. 12, No. 6, 1948.

2. Kolodner, I.I., Free boundary problem for the heat equation with applications of change of phase. *Comm. Pure Appl. Math.* Vol. 9, No. 1, 1956.

3. Slezkin, N.A., *Dinamika viazkoi neszhimaemoi zhidkosti (Dynamics of a viscous incompressible fluid)*. Gostekhteoretizdat, Moscow, 1955.

4. Safronchik, A.I., Neustanovivsheesia techenie viazko-plastichnogo materiala mezhdu parallel'nymi stenkami (Non-steady flow of a visco-plastic material between parallel walls). *PMM* Vol. 23, No. 5, 1959.

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